Dot and Cross Products

Cross Product

Definition 1: The cross product of two vectors $\underline{a}$ and $\underline{b}$ is a vector which is orthogonal to both $\underline{a}$ and $\underline{b}$ whose magnitude is given by the area of the parallelogram defined by $\underline{a}$ and $\underline{b}$ and whose direction is given by the right hand rule (cork screw rule) or by comparison with the standard x,y,z axis.

Definition 2 $\underline{a} \times \underline{b} = |a||b| \sin \vartheta \underline{\hat{n}}$ where θ is the angle between $\underline{a}$ and $\underline{b}$ and $\underline{\hat{n}}$ is a unit magnitude vector orthogonal to $\underline{a}$ and $\underline{b}$ in the sense given by the right hand rule.

Useful Mathematical Relations:

$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$ - cross products anti-commute (think right hand rule)

$\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$ - cross products are distributive (note the order is still important)

$(\underline{a} \times \underline{b}).\underline{a} = (\underline{a} \times \underline{b}).\underline{b} = 0$ - which should be obvious

Applications of Cross Product

The force ($\underline{F}$) on a segment of a wire carrying a current I whose ends are joined by the vector $\underline{L}$ in a uniform magnetic field ($\underline{B}$) is given by

$$\underline{F} = I\,\underline{L} \times \underline{B}$$ - ordering is important

Magnetic field ($\underline{B}$) due to a charge q moving with velocity $\underline{v}$ at point whose position relative to the charge is $\underline{r}$ is given by

$$\underline{B} = \frac{\mu_0}{4\pi} \frac{q\underline{v} \times \underline{r}}{r^2}$$

The angular momentum of a particle ($\underline{L}$) with momentum $\underline{p}$ and position $\underline{r}$ relative to the position at which the angular momentum is measured is given by

$$\underline{L} = \underline{r} \times \underline{p}$$

The torque $\underline{\tau}$ due to a force $\underline{F}$ applied to a particle at position $\underline{r}$ relative to the position at which the torque is measured is given by

$$\underline{\tau} = \underline{r} \times \underline{F}$$

A plane in space can be defined in many ways. One obvious way is to define one point on the plane $\underline{c}$ and two vectors $\underline{a}$ and $\underline{b}$, which are not parallel, in the plane. A point is on the plane if the difference between its position $\underline{r}$ and the known point on the plane can be written as a sum of the two vectors in the plane. A much algebraically cleaner method of writing the equation is given by

$$(\underline{r} - \underline{c}).\underline{n} = 0 \text{ where } \underline{n} \text{ is normal to the plane i.e. } \underline{n} = \underline{a} \times \underline{b}$$

Diagram needed here

Problems

Cross Products

Question 1

A short segment of wire whose ends are joined by the vector $\underline{L}$ = (1,2,0) cm is carrying a current of 1 A in the presence of a uniform magnetic field $\underline{B}$ = (0,0,2) T what is the force due to the magnetic field on the wire?

Question 2

a) Two particles at positions $\underline{r}$= (1,1,0) mm and $-\underline{r}$ are acted on by forces $\underline{F}$ = (1,-1,0) N and $-\underline{F}$; what is the total to torque on the two particles measured at the origin?
b) Two particles at positions $\underline{r}$= (1,1,0) mm and $-\underline{r}$ are acted on by forces $\underline{\mathrm{F}}$ = (1,-1,0) N and $\underline{\mathrm{F}}$; what is the total to torque on the two particles measured at the origin?

Question 3

Work out the cross product of vectors $\underline{a}$ = (1,-2,4) and $\underline{b}$ = (-3,-3,2) by writing out both vectors as a sum of the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$, and using the distributive and commutative rules for cross products.

Question 4

a) The sides of a parallelogram are defined by the vectors $\underline{a}$ = (1,2,3) and $\underline{b}$ = (2,1,0). Calculate a vector normal to the surface of the parallelogram whose magnitude is equal to the area of the parallelogram.
b) A wire is formed into the shape of the parallelogram and place in a flow of air whose velocity $\underline{v}$ = (1,1,1) ms^{-1}. What volume of air passes through the parallelogram in one second? (Note: This part of the question is much harder and will require you to use the dot product).

Written by David Smith, edited by BSL, 2007